

The effect of structural aspect for planar systems with 2DOF upon the stability of motion

S Alaci¹, F-C Ciornei¹, C-C Suci¹ and I-C Romanu¹

¹Mechanics and Technologies Department, „Stefan cel Mare” University of Suceava, Suceava, Romania

E-mail: stelian.alaci@usm.ro

Abstract. The paper analyses all planar chains with 2DOF from the point of view of stability of motion. For the rotation-prismatic structural solution first there are obtained the equations of motion and then, the numerical integration procedure is applied. A strong instability of the system can be noticed. The same dynamical system is modelled using dynamical analysis software and the instability is confirmed.

1. Introduction

There are uncommon the cases when a part of a mechanical structure moves unrestrained, without interacting with other parts from its proximity. The majority of the mobile mechanical structures are made of solid elements that come into contact to each other. When two of the components of a system make contact the number of degrees of freedom of the two elements reduces. One of the main classification criteria of the constraints which take place between two parts, named kinematical joints, is the class of the pair, defined as the number of degrees of freedom taken for one of the element of the pairs, when the other element is considered fixed, [1]. Considering the fact that a rigid body has 6 degrees of freedom (DOF) it results that the class of the pair can have values from 1 to 5, the limit case of 0 representing the lack of linkage and the pair of class 6 represents the relative rigidity of the two parts that behave as a single part.

The second especially important notion when analyzing the mobility of a structure is the family, concept that illustrates the number of common restraints imposed to each element of the structure. The monographs of mechanisms theory consider as main categories of mechanisms: spatial mechanisms, in the general significance, spherical mechanisms that have as characteristic the existence of the same fixed point for all the elements of the mechanism and planar mechanisms that have as characteristic the fact that all the elements move parallel to the same plane. Both spherical and planar mechanisms are mechanisms of family 3, [2]. Since a kinematical chain of a certain family cannot contain pairs of a class inferior or equal to the family of the chain, for a planar kinematical chain - that will be the referred next, there is possible the existence of pairs of class 5 - rotation R and prismatic T and also of superior pairs of class 4. The dynamical systems with two degrees of freedom are the simplest systems where the chaos phenomenon can appear.

2. Possible structural solutions for planar kinematical chains with 2 DOF

The effect of chaos for a double physical pendulum, figure 1, is presented in [3] for the situation when the launch position differs substantially from the position of static equilibrium, that corresponds to the vertical orientation of both rods and with the center of mass in the lowest position. To illustrate the

effect of chaos that occurs in the system, two identical pendula were considered, positioned symmetrically with respect to a vertical plane. The motion of the pendula was modeled using dynamical simulation software. It was observed that for a relatively short time the motions of the pendula were identical, but after that, the oscillations of the pendula differed. From experiments performed with the double pendulum, figure 1, it was concluded that the attempt to obtain identical motions from two similar launches, with the same angular amplitudes, $\theta_{10} \geq 90^\circ$, $\theta_{20} \geq 90^\circ$, is unsuccessful. The explanation of the phenomenon resides in the strong dependency of the evolution of the system on the initial conditions. But, in spite of the efforts, the precision of obtaining the initial position cannot be superior to the precision of the instruments of measure used in the estimation of the positional parameters of the system.

In figure 2 is presented a system having in structure a prismatic pair and a rotation pair for which Voinea [4] presents the differential equations of motion. In order to appreciate how stable the system is, the manner used for the double pendulum is applied. For that reason, using dynamical simulation software, two identical systems were modeled, positioned in mirror, with the same initial position, figure 4. The angle characterizing the initial position of the rod is $\theta_0 = 135^\circ$. Unlike the double pendulum from figure 1, where after a few seconds the motions of the pendula are completely different, in this case, the motions of the two systems are rigorously identical, as shown in figure 5 where the variations of velocities of the prismatic bodies were plotted. To validate this affirmation, in figure 6 is presented the dependency of the velocity of prism 2 as a function of the velocity of the prism 1. The straight line plot from figure 6 shows the equality of the two velocities.

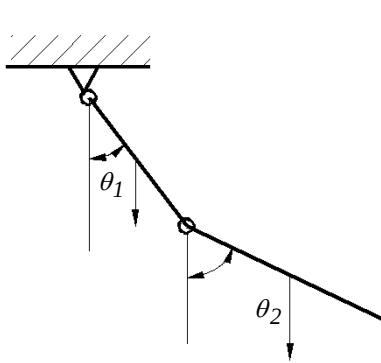


Figure 1. Kinematical chain 2 DOF with two rotation pairs

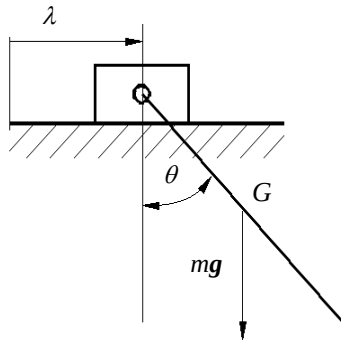


Figure 2. Kinematical chain 2 DOF with one rotation pair and one prismatic pair

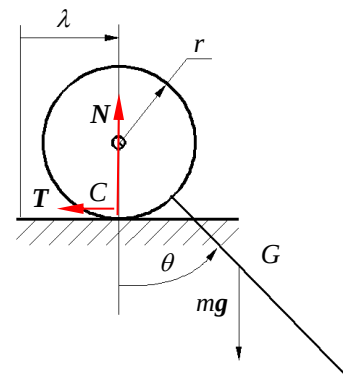


Figure 3. Kinematical chain 2 DOF with higher pair

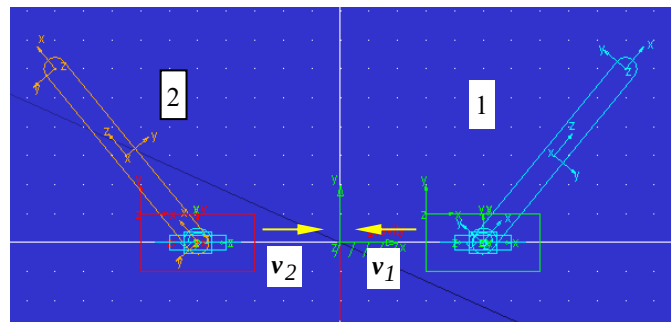


Figure 4. Mirrored identical systems

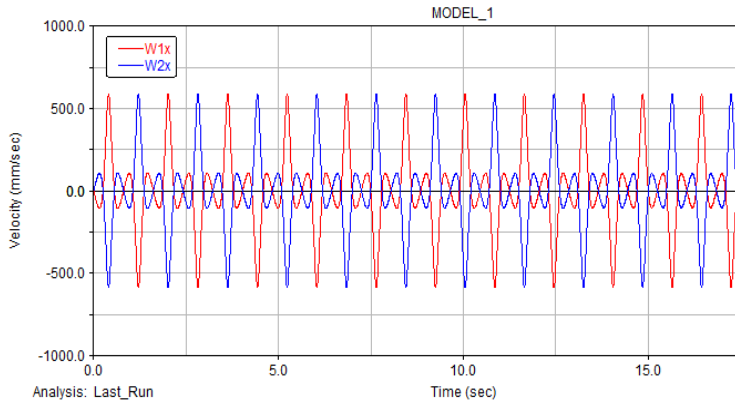


Figure 5. Variation of the velocities of the prisms

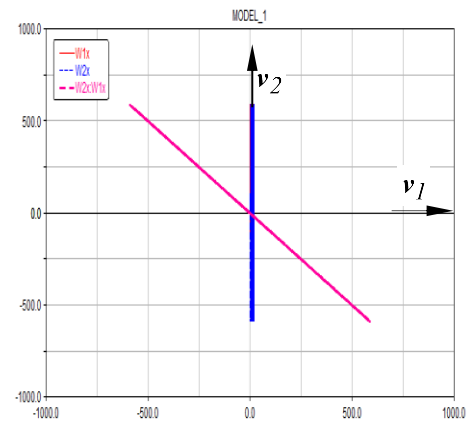


Figure 6. The dependency v_2 vs v_1

A kinematical chain that contains a higher kinematic pair is presented in figure 3. The system has two degrees of freedom λ, θ because when the friction is absent, the fundamental theorems of dynamics provide three scalar equations (the momentum theorem gives two scalar equations and the moment of momentum theorem gives a single equation) which allow for finding three unknowns, here the two positional parameters λ and θ and the magnitude of the normal reaction N . But particularly interesting is the case when friction occurs in the higher pair. The magnitude of the friction force T is added besides the three unknowns λ, θ and N . To solve the problem, a supplementary equation is required. Two cases can be distinguished, depending on the friction type:

a) when rolling friction is present, the supplementary equation is $\dot{\lambda} = r\dot{\theta}$ and the system has 1 DOF, and in fact it is a cycloidal pendulum with the dynamics presented in [5] for which there is no risk of chaos phenomenon occurrence.

b) when dry friction is present, the fourth equation is given by the relation between the magnitude of friction force and the magnitude of normal reaction, $T = \mu N$ according to Coulomb's law. In this case the system has again 2 DOF.

3. The 2DOF linkage with rotation and prismatic pairs

Next it is presented the effect produced by reciprocal changing the pairs from the kinematical chain from figure 2, as in figure 7, problem proposed by Spiegel [6]. It is considered the particular case when the homogenous rod is jointed to the ground in the center of mass. A sleeve with negligible dimensions slides without friction on the rod. A

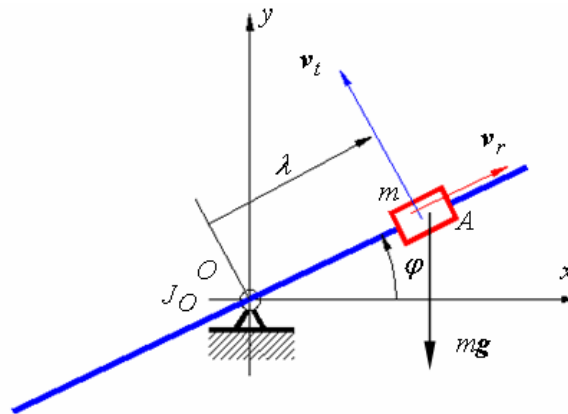


Figure 7. System with 2 DOF

The rod of length L and diameter d has the joint in the center of mass O , and can rotate about an horizontal axis. The moment of inertia of the bar with respect to the axis of rotation is J_O . The position of the rod is specified using the angle φ made with the vertical direction. A sleeve of negligible dimensions, but of mass m is placed on the rod, having the position specified by the distance λ from the joint of the bar to the center of the sleeve. The hypothesis of ideal linkages is accepted and therefore the motion of the system can be found using the Lagrange equations of the second kind [7]:

$$\frac{d}{dt} \frac{\partial E_c}{\partial \dot{q}_k} - \frac{\partial E_c}{\partial q_k} = Q_k \quad (1)$$

where E_c is the kinetic energy of the system, q_k are the generalized coordinates and Q_k are the generalized forces. In this case the generalized coordinates are the angle φ and the distance λ . The kinetic energy of the system consists in the kinetic energy of rotation of the rod:

$$E_{C_1} = \frac{1}{2} J_O \dot{\varphi}^2 \quad (2)$$

and the translation energy of the sleeve:

$$E_{C_2} = \frac{1}{2} m v^2 \quad (3)$$

As noticed from figure 7, the absolute velocity of the sleeve is the vector sum between the transport velocity $\mathbf{v}_t$ and the relative velocity $\mathbf{v}_r$, the two components being perpendicular: the first is normal to the rod and the second is oriented along the rod.

$$\mathbf{v} = \mathbf{v}_t + \mathbf{v}_r \quad (4)$$

The magnitude of the transport velocity is:

$$v_t = \lambda \dot{\varphi} \quad (5)$$

and of the relative velocity is:

$$v_r = \dot{\lambda} \quad (6)$$

The final form of the kinetic energy of the system is:

$$E_c = \frac{1}{2} J_O \dot{\varphi}^2 + \frac{1}{2} m (\lambda^2 \dot{\varphi}^2 + \dot{\lambda}^2) \quad (7)$$

The single force that produces mechanical work in the system is the force of gravity of the sleeve.

$$\mathbf{G} = -m\mathbf{g} = -mg\mathbf{j} \quad (8)$$

where g is the gravitational acceleration. The position vector of the center of mass of the sleeve is:

$$\mathbf{r} = \lambda(\mathbf{i} \cos \varphi + \mathbf{j} \sin \varphi) \quad (9)$$

The elementary variation of the position of the center of mass is:

$$\delta \mathbf{r} = (\mathbf{i} \cos \varphi + \mathbf{j} \sin \varphi) \delta \lambda + \lambda(-\sin \varphi \mathbf{i} + \cos \varphi \mathbf{j}) \lambda \delta \varphi \quad (10)$$

The virtual work of the mass is:

$$\delta L = \mathbf{G} \cdot \delta \mathbf{r} = -mg \sin \varphi \delta \lambda - \lambda mg \cos \varphi \delta \varphi \quad (11)$$

For the generalized coordinate φ it is obtained:

$$\frac{\partial E_c}{\partial \dot{\varphi}} = J_O \dot{\varphi} + m\lambda^2 \dot{\varphi} \quad (12)$$

$$\frac{d}{dt} \frac{\partial E_c}{\partial \dot{\varphi}} = J_O \ddot{\varphi} + m\lambda^2 \ddot{\varphi} + 2m\lambda \dot{\varphi} \dot{\lambda} \quad (13)$$

$$\frac{\partial E_c}{\partial \varphi} = 0 \quad (14)$$

The generalized force Q_φ is:

$$Q_\varphi = \frac{\delta L}{\delta \varphi} = -mg\lambda \sin \varphi \quad (15)$$

For the generalized coordinate λ it is obtained:

$$\frac{\partial E_c}{\partial \dot{\lambda}} = m\dot{\lambda} \quad (16)$$

$$\frac{d}{dt} \frac{\partial E_c}{\partial \dot{\lambda}} = m\ddot{\lambda} \quad (17)$$

$$\frac{\partial E_c}{\partial \lambda} = m\lambda \dot{\varphi}^2 \quad (18)$$

The generalized force Q_λ is:

$$Q_\lambda = \frac{\delta L}{\delta \lambda} = -mg \sin \varphi \quad (19)$$

The motion of the system is described by the system of differential equations:

$$\begin{cases} (J_O + m\lambda^2) \ddot{\varphi} + 2m\lambda \dot{\varphi} \dot{\lambda} = -mg\lambda \cos \varphi \\ m\ddot{\lambda} - m\lambda \dot{\varphi}^2 = -mg \sin \varphi \end{cases} \quad (20)$$

The system (20) is written under the final form:

$$\begin{cases} \ddot{\varphi} = -\lambda \frac{2\dot{\varphi}\dot{\lambda} + g \cos \varphi}{\lambda^2 + J_O / m} \\ \ddot{\lambda} = \lambda \dot{\varphi}^2 - g \sin \varphi \end{cases} \quad (21)$$

In order to integrate the system (20), two new variables are introduced:

$$u = \dot{\varphi} \quad (22)$$

$$v = \dot{\lambda} \quad (23)$$

Now, using the above notations, the system (20) can be written in matrix form:

$$\begin{bmatrix} \dot{\varphi} \\ \dot{\lambda} \\ \dot{u} \\ \dot{v} \end{bmatrix} = \begin{bmatrix} u \\ v \\ -\frac{2uv + g \cos \varphi}{J_O / m + \lambda^2} \lambda \\ \lambda u^2 - g \sin \varphi \end{bmatrix} \quad (24)$$

The system (24) is a system of differential equations of first order with four unknowns and it was written under this form because it allows for applying the Runge-Kutta [8] integration methodology. In order to integrate it, it is required the stipulation of the initial conditions. Explicitly, the values of the functions φ , λ , u and v must be specified for the initial moment. Aiming to complete a comparison with an experimental test, it is considered that the system is initially at rest:

$$u = 0, \quad v = 0 \quad (25)$$

and only the position parameters for the initial moment, must be specified:

$$\varphi = \varphi_0, \quad \lambda = \lambda_0 \quad (26)$$

It was avoided the case of a system in motion for the initial conditions because in practical tests it is difficult to ensure an imposed initial motion to a system.

For arbitrary values of the inertial characteristics of the rod and sleeve, the motion of the sleeve is easy to guess, as due to own weight it slides along the rod and at a certain time it leaves the rod. In the meantime, the rod performs a rotation motion of amplitude 2-3 rad. The trajectory of the center of mass of the sleeve is presented in figure 8a-b, obtained after integration of the equations (22), where the characteristics considered are: $L = 1m$, $d = 0.01m$, $J_O = 0.051050880620834 kg \cdot m^2$, $m = 0.54870382564081kg$, $\lambda_0 = L/2$, $\theta_0 = \pi/3$, $\dot{\lambda}_0 = 0$, $\dot{\theta}_0 = 0$.

In order to highlight the instability of the solution, in figure 8b it is represented the trajectory of the center of mass obtained for the same conditions, but for $\theta_0 = 0.99999\pi/3$.

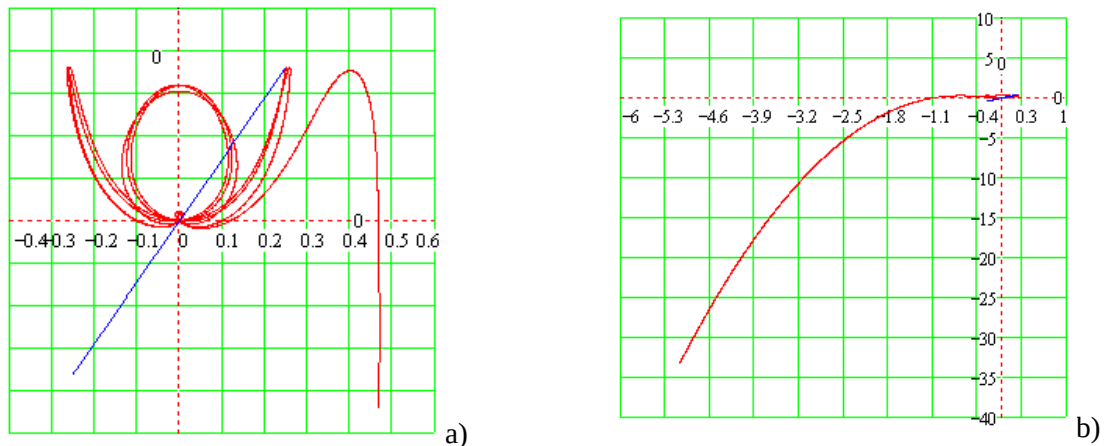


Figure 8. The trajectory of the center of mass of the sleeve

The same conclusion was reached after modelling the behaviour of the system using the dynamic simulation software, as presented in figure 9.

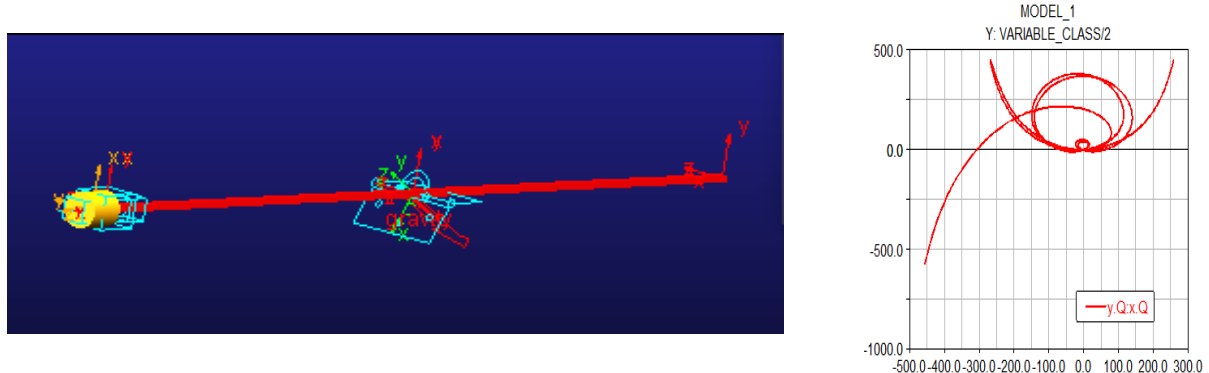


Figure 9. Simulation of the motion of the system using specialized software

4. Conclusions

The paper examines all kinematical planar chains with 2 DOF from the point of view of motion stability. There are four possible structural solutions: rotation-rotation, prismatic-rotation, rotation-prismatic and higher pair linkage, from which the first two are classical examples in dynamical systems monographs and the last two solutions are seldom met – most probable due to the deficiency of practical usage.

Though for the structural possible solutions that have lower pairs the friction from the pairs was neglected, for the system with higher pair, the effects of friction were considered because the type of friction (rolling or sliding) defines the DOF of the system (one or two, respectively) which is essential upon the stability of the motion of the system. For the rotation-prismatic 2DOF system with the rotation pair placed on the ground, the equations of motion were obtained applying the Lagrange's equations of second kind. The system of differential equations was numerically integrated and a strong instability of the obtained solution was highlighted. Actually, when altering any of the initial conditions with a fraction of $10^{-4}\%$, a complete different law of motion of the system is obtained.

The same conclusion was reached after studying the system with specialised software for dynamical analyses.

The paper underlines the necessity of correct structural design, of large interest for practical devices and it also is suitable for didactical activity in exemplification of simple yet very instable dynamical systems.

References

- [1] Uicker J, Pennock G and Shigley J 2010 *Theory of Machines and Mechanisms* (Oxford UP)
- [2] McCarthy J M 1990 *Introduction in Theoretical Kinematics* (MIT Press) p 130
- [3] Alaci S, Alexandru C, Ciornei F-C, Doroftei I and Irimescu L 2020 Chaos illustrations in dynamics of mechanisms *Mechanisms and Machine Science* **89** pp 297-304
- [4] Voinea R P, Voiculescu D and Ceașu V 1983 *Mechanics* 2nd Ed (EDP București) p 615
- [5] Ciornei M C, Alaci S, Ciornei F C and Romanu I C 2017 A method for the determination of the coefficient of rolling friction using cycloidal pendulum *IOP Conference Series: MSE* **227** (1) 012027
- [6] Spiegel M, 1980 *Schaum's outline of theory and problems of theoretical mechanics: With an introduction to Lagrange's equations and Hamiltonian theory* (McGaw Hill) p 368
- [7] Ardema M 2005 *Analytical Dynamics* (Springer US) p 340
- [8] Hermann M and Saravi M 2014 *A First Course in Ordinary Differential Equations: Analytical and Numerical Methods* (Springer India) p 288